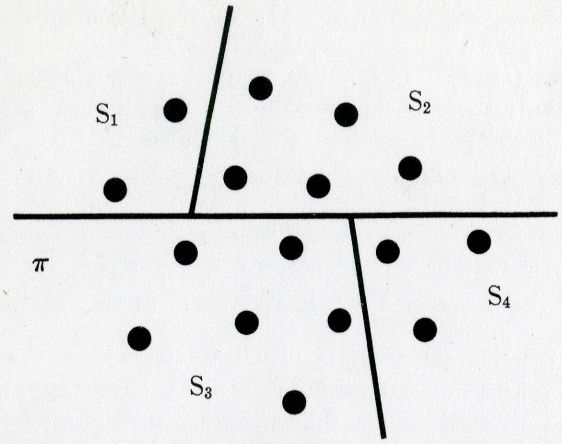

The information contained in M is $H(M)$. The information contained in the S_α taken separately is $\sum_\pi H(S_\alpha)$. Except in the case where there is no interaction at all between the different subsystems, the second of these two expressions will be larger than the first, because some information will, as it were, be counted more than once. As a result, we may use the difference between the two expressions, $\left\{\left[\sum_\pi H(S_\alpha)\right] - H(M)\right\}$ as a measure of the strength of the connections severed by the partition π.[12] The larger it is, the stronger the connections severed are. The smaller it is, the weaker the connections are, and the less information transfer there is across the partition. The value of this difference is given by

$$\left\{ (s_1 + \cdots + s_\mu) \log 2 + \frac{\delta^2}{2} \sum_{S_1,S_2,\cdots} \nu_{ij}^2 - m \log 2 - \frac{\delta^2}{2} \sum_M \nu_{ij}^2 \right\},$$

where the sum $\sum_{S_1,S_2,\cdots}$ is taken only over pairs i,j, which are wholly contained in one of the S_α. The difference, or redundancy, of the partition is therefore $\frac{1}{2}\delta^2 \sum_\pi \nu_{ij}^2$, where the sum is taken over all links ij cut by the partition π.

As it stands the redundancy $\frac{1}{2}\delta^2 \sum_{\pi} \nu_{ij}^2$ does not give us a fair basis for comparison of different π. Each π belongs to a certain "partition-type." That is, the subsets it defines have $s_1, s_2, \cdots s_\mu$ variables respectively, and the collection of numbers $\{s_1, s_2, \cdots s_\mu\}$ defines the partition-type. The value of $\sum_{\pi} \nu_{ij}^2$ will tend to be lower for some partition-types than others.

To normalize the redundancy, we now compute the expected value and variance of $\sum_{\pi} \nu_{ij}^2$ as a function of the partition-type, given a random distribution of l links among the $\frac{1}{2}m(m-1)$ possible spaces for links provided by m vertices. (For the sake of simplicity we shall assume that no space can hold more than one link, i.e., $\nu = 1$, so that $\nu_{ij} = 0$ or 1).[13] If all distinguishable distributions of the l links are equiprobable, the expected value and variance of $\sum_{\pi} \nu_{ij}^2$ will depend on four parameters. Two of them are constant. The first, l, is the number of links in L. The second, l_0, is the number of possible spaces to which links might be assigned. It is given by $l_0 = \frac{m(m-1)}{2}$. The other two parameters depend on the partition π. The first, l_0^π, is the number of the l_0 potential spaces which are cut by the partition π, i.e., the number of vertex pairs in which vertices come from different subsets of the partition. This depends on the partition-type of π, and is given by $l_0^\pi = \sum_{\pi} s_\alpha s_\beta$, where s_α is the number of variables in S_α. We note that $l_0^\pi \leqslant l_0$. The second of these parameters, l^π, is the number of actual links cut by the partition π. This is given by $l^\pi = \sum_{\pi} |\nu_{ij}|$. Of course $l^\pi \leqslant l$.

We consider first the expected value of $\sum_{\pi} \nu_{ij}^2 = \mathrm{E}\left(\sum_{\pi} \nu_{ij}^2\right)$. Since the ν_{ij} are independent we may write

$$\mathrm{E}\left(\sum_{\pi} \nu_{ij}^2\right) = \sum_{\pi} \mathrm{E}(\nu_{ij}^2) = l_0^\pi \mathrm{E}(\nu_{ij}^2),$$

where $E(\nu_{ij}^2)$ is the expected value of ν_{ij}^2 for some one fixed space spanning two points i,j.

Clearly $$E(\nu_{ij}^2) = \frac{l}{l_0},$$
so this reduces to
$$E\Big(\sum_{\pi} \nu_{ij}^2\Big) = \frac{l l_0^{\pi}}{l_0},$$
which depends on the value of l_0^{π} and so on the partition-type of π.

Let us now consider the variance of $\sum_{\pi} \nu_{ij}^2$.[14]

$$\mathrm{Var}\,\Big(\sum_{\pi} \nu_{ij}^2\Big) = E\Big[\Big(\sum_{\pi} \nu_{ij}^2\Big)^2\Big] - \Big[E\Big(\sum_{\pi} \nu_{ij}^2\Big)\Big]^2.$$

We already know the value of the second term. As for the first:

$$E\Big[\Big(\sum_{\pi} \nu_{ij}^2\Big)^2\Big] = E\Big[\sum_{\pi} \nu_{ij}^4 + 2\sum_{\pi} \nu_{ij}^2 \nu_{kl}^2\Big].$$

Since we have arranged to take ν_{ij} as positive, $=0$ or 1, we have $\nu_{ij}^4 = \nu_{ij}^2 = \nu_{ij}$ and hence:

$$\mathrm{Var}\,\Big(\sum_{\pi} \nu_{ij}^2\Big) = E\Big(\sum_{\pi} \nu_{ij}\Big) + 2E\Big(\sum_{\pi} \nu_{ij}\nu_{kl}\Big) - \Big[E\Big(\sum_{\pi} \nu_{ij}\Big)\Big]^2.$$

Let us consider two fixed spaces ij and kl.

Now

$$\begin{aligned} E(\nu_{ij}\nu_{kl}) &= 0 \cdot p(\nu_{ij}\nu_{kl} = 0) + 1 \cdot p(\nu_{ij}\nu_{kl} = 1) \\ &= p(\nu_{ij}\nu_{kl} = 1) \\ &= \frac{l}{l_0} \cdot \frac{l-1}{l_0 - 1} = \frac{l(l-1)}{l_0(l_0-1)}. \end{aligned}$$

$$\begin{aligned} \therefore \quad E\Big(\sum_{\pi} \nu_{ij}\nu_{kl}\Big) &= \tfrac{1}{2} l_0^{\pi}(l_0^{\pi} - 1) \cdot E(\nu_{ij}\nu_{kl}) \\ &= \tfrac{1}{2} l_0^{\pi}(l_0^{\pi} - 1) \cdot \frac{l(l-1)}{l_0(l_0-1)}. \end{aligned}$$

This gives us

$$\mathrm{Var}\,\Big(\sum_{\pi} \nu_{ij}^2\Big) = \frac{l \cdot l_0^{\pi}}{l_0} + l_0(l_0^{\pi} - 1)\,\frac{l(l-1)}{l_0(l_0-1)} - \Big(\frac{l \cdot l_0^{\pi}}{l_0}\Big)^2$$